ON THE INVARIANTS OF QUADRATIC DIFFERENTIAL FORMS, II*

BY

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In the following paper I propose to extend to differential parameters the work of an earlier article † on the determination of the number of differential invariants of quadratic differential forms in n variables.

§ 1. *Introduction.*

The form in question,

$$\phi \equiv \sum_{i=1}^{n} \sum_{k=1}^{n} a_{ik}(x_1, \cdots, x_n)\, dx_i\, dx_k,$$

and the transformation to which it is subjected,

$$Xf \equiv \sum_{r=1}^{n} \xi_r(x_1, \cdots, x_n) \frac{\partial f}{\partial x_r},$$

are the same as in the earlier paper.

A *differential parameter* of the form ϕ is a function which is unaltered by any transformation of the group, and which may involve the following quantities:

a) The coefficients a_{ik} and their derivatives of various orders with respect to the variables x;

b) certain arbitrary functions $U(x_1, \cdots, x_n)$, numerically unchanged by the group (2), and the various derivatives of these functions with respect to the variables x;

c) if the variables x are not all independent, the derivatives of those which are dependent with respect to those which are independent.

If the differential parameter does not contain the functions U, or their derivatives, or the derivatives of the x's, it will be called a *differential invariant* (Gaussian invariant).

If the differential parameter does not contain the derivatives of the x's it will be called a *differential parameter of the first type* (Beltramian invariant).

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† HASKINS, *On the invariants of quadratic differential forms.* Transactions of the American Mathematical Society, vol. 3 (1902), p. 71. (Cited hereafter as *Invariants* I.)

If the differential parameter contains the derivatives of the x's, but does not contain the U's or their derivatives, it will be called a *differential parameter of the second type* (Minding invariant).

The *order μ of a differential invariant* is the order of the highest derivative of any of the a_{ik}'s appearing in it.

The *order μ of a differential parameter* is the order of the highest derivative of any of the U's or x's appearing in it.

We shall find that differential *parameters* of a given order μ are naturally associated with differential *invariants* of the next lower order $\mu - 1$.

Differential parameters of the second type are always expressible in terms of differential parameters of the first type. *

For suppose the x's are not independent, but are bound by relations

$$U_i(x_1, \cdots, x_n) = 0 \qquad (i = 1, 2, \cdots, n - m),$$

and let F be a differential parameter of the second type. By means of the equations $U = 0$ and their derivatives with respect to m of the variables x, taken as independent, we may express in terms of the derivatives of the functions U the derivatives of the dependent x's with respect to the independent x's. When these expressions are substituted in F it becomes a function of the partial derivatives of the U's. It has the property of invariance and is therefore a parameter of the *first type*.

Geometrically, parameters of the second type express invariant properties of spaces of lower dimensions drawn in the space whose linear element is given by ϕ, e. g., invariant properties of curves on surfaces.

Not all parameters of the first type are also of the second type. Hence it is desirable to investigate the parameters of the second type as well as those of the first.

The differential parameters are solutions of certain complete systems of linear partial differential equations. The number of parameters is therefore equal to the excess of the number of variables in these equations over the number of independent equations. The computation of the number of equations and of the number of variables is very simple. Consequently the difficulty of the problem lies in the determination of the independence of the equations in question. It will be found that this determination can, save in a few cases, be referred to the corresponding problem in the case of the invariants. In fact there remain but five cases which cannot be so referred, viz.:

$$n = 2, \qquad \mu = 1, 2, 3;$$

$$n > 2, \qquad \mu = 1, 2.$$

* This point, apparently unnoticed by ŻORAWSKI, has recently been emphasized by FORSYTH, Philosophical Transactions, vol. 201 (1903), A, p. 333.

The cases $n = 2$ have been fully treated by ŻORAWSKI,* and may be omitted here. We have to consider, then, only the cases $n > 2$; $\mu = 1, 2$. It may be noted that LEVI-CIVITA has obtained a lower limit for the number of parameters.†

The consideration of the cases $n > 2$, $\mu = 1, 2$, and $\mu > 2$ occupies the last five sections of the article. The equations determining the parameters are so numerous and complicated that, in order to prove their independence, it is found desirable to introduce certain new variables, in particular, the three index symbols of CHRISTOFFEL and the covariant derivatives of RICCI. In order not to interrupt the course of the reasoning by which the independence of the equations is shown, the definitions of these new variables and the discussion of such of their properties as are used later have been collected in a preliminary section (§ 2). Since in each case the proof depends on the non-*identical* vanishing of certain determinants, it is found that the work is facilitated by the specialization of the form and of the variables.

The results of the investigation may be expressed in the following

THEOREMS:

I. *The number of parameters of order* μ *and of the first type* (*Beltramian invariants*) *is*

$$(a)\quad \begin{cases} \dfrac{m(m+1)}{2}, & \text{if } \mu = 1, \text{ and } m \leqq n; \\[2ex] mn - \dfrac{n(n-1)}{2}, & \text{if } \mu = 1, \text{ and } m \geqq n; \end{cases}$$

$$(b)\quad \begin{cases} \dfrac{mn(n+1)}{2} - \dfrac{(n-m)(n-m-1)}{2}, & \text{if } \mu = 2, \text{ and } m \leqq n; \\[2ex] \dfrac{mn(n+1)}{2}, & \text{if } \mu = 2, \text{ and } m \geqq n; \end{cases}$$

$$(c)\quad \frac{mn(n+1)(n+2)}{6} + \frac{n(n-1)}{2}, \qquad \text{if } \mu = 3;$$

$$(d)\quad \frac{m(n+\mu-1)!}{(n-1)!\,\mu!}, \qquad \text{if } \mu \geqq 4;$$

where n = *number of variables* x, m = *number of functions* U.

II. *The number of parameters of order* μ *and of the second type* (*Minding invariants*) *is*

$$(a)\qquad 0, \qquad \text{if } \mu = 1;$$

* ŻORAWSKI, *Ueber Biegungsinvarianten*, Acta Mathematica, vol. 16 (1892), p. 41.

† LEVI-CIVITA, *Sugli invarianti assoluti*, Atti d. R. Ist. Veneto, ser. 7, vol. 52 (1894), p. 1498.

$$(b)\quad \begin{cases} \dfrac{m(m+3)(m^2-m+2)}{8}, & \textit{if } \mu = 2, \textit{ and } \dfrac{m(m+3)}{2} \leqq n; \\ (n-m)\left\{m+\dfrac{m(m+1)}{2}\right\}-\dfrac{n(n-1)}{2}, & \textit{if } \mu = 2, \textit{ and } \dfrac{m(m+3)}{2} \geqq n; \end{cases}$$

$$(c)\quad \begin{cases} (n-m)\dfrac{m(m+1)(m+2)}{6}+\dfrac{m(m+3)}{2}\left\{n-\dfrac{(m+1)(m+2)}{4}\right\}, & \\ & \textit{if } \mu = 3, \textit{ and } \dfrac{m(m+3)}{2} \leqq n; \\ (n-m)\dfrac{m(m+1)(m+2)}{6}+\dfrac{n(n-1)}{2}, & \\ & \textit{if } \mu = 3, \textit{ and } \dfrac{m(m+3)}{2} \geqq n; \end{cases}$$

$$(d)\quad (n-m)\frac{(m+\mu-1)!}{(m-1)!\,\mu!}, \qquad \textit{if } \mu \geqq 4;$$

where n = number of variables x, $n-m$ = number of fixed relations among the variables x.

§ 2. *Differential equations of the problem.*

As in the case of the invariants the group must be "extended" to include the functions involved in the parameters sought.

In dealing with parameters of the second type we may regard $n-m$ of the variables x as functions of the remaining m and use the transformation

$$(1)\qquad Xf \equiv \sum_{r=1}^{n} \xi_r(x_1, \cdots, x_n)\frac{\partial f}{\partial x_r},$$

or we may regard them all as functions of m new independent variables $z_1, \cdots, z_m$; the system of variables $x_1, \cdots, x_n$; $z_1, \cdots, z_m$, being subjected to the transformation

$$(2)\qquad Yf \equiv \sum_{r=1}^{n} \xi(x_1, \cdots, x_n)\frac{\partial f}{\partial x_r} + \sum_{r=1}^{m} \zeta_r(z_1, \cdots, z_m)\frac{\partial f}{\partial z_r}.$$

The following notation will be used.

$$a_{ik} \equiv \text{coefficient of } dx_i dx_k \text{ in } \phi,$$

$$p_{ik} \equiv \frac{\partial f}{\partial a_{ik}},$$

$$\underset{l_1 \ldots l_\mu}{a_{ik}} \equiv a_{ik|l_1 \cdots l_\mu} \equiv \frac{\partial^\mu a_{ik}}{\partial x_{l_1} \cdots \partial x_{l_\mu}},$$

$$p_{\substack{ik\\ l_1\dots l_\mu}} \equiv p_{ik/l_1\dots l_\mu} \equiv \frac{\partial f}{\partial a_{ik/l_1\dots l_\mu}},$$

$$a \equiv |a_{11}\cdots a_{nn}| \equiv \text{discriminant of the form } \phi,$$

$$A_{ik} \equiv \frac{1}{a}\{\text{cofactor of } a_{ik} \text{ in } a\},$$

$$P_{ik} \equiv \frac{\partial f}{\partial A_{ik}},$$

(3) $$\begin{bmatrix} ik \\ l \end{bmatrix} \equiv \frac{1}{2}(a_{\substack{kl\\ i}} + a_{\substack{il\\ k}} - a_{\substack{ik\\ l}}) \equiv \text{CHRISTOFFEL's three-index symbol of the first kind},$$

$$q_{\substack{ik\\ l}} \equiv q_{ik/l} \equiv \frac{\partial f}{\partial \begin{bmatrix} ik \\ l \end{bmatrix}},$$

(4) $$\begin{Bmatrix} ik \\ l \end{Bmatrix} \equiv \sum_{\nu=1}^{n} A_{\nu l}\begin{bmatrix} ik \\ \nu \end{bmatrix} \equiv \text{CHRISTOFFEL's three-index symbol of the second kind},$$

$$\xi_{\substack{r\\ l_1\dots l_\mu}} \equiv \xi_{r/l_1\dots l_\mu} \equiv \frac{\partial^\mu \xi_r}{\partial x_{l_1}\cdots\partial x_{l_\mu}},$$

$$\zeta_{\substack{r\\ l_1\dots l_\mu}} \equiv \zeta_{r/l_1\dots l_\mu} \equiv \frac{\partial^\mu \zeta}{\partial z_{l_1}\cdots\partial z_{l_\mu}},$$

$$U_{\substack{h\\ l_1\dots l_\mu}} \equiv U_{h/l_1\dots l_\mu} \equiv \frac{\partial^\mu U_h}{\partial x_{l_1}\cdots\partial x_{l_\mu}},$$

$$Q_{\substack{h\\ l_1\dots l_\mu}} \equiv Q_{h/l_1\dots l_\mu} \equiv \frac{\partial f}{\partial U_{h/l_1\dots l_\mu}},$$

(5) $$V_{\substack{h\\ ik}} \equiv V_{h/ik} \equiv U_{\substack{h\\ ik}} - \sum_{s=1}^{n}\begin{Bmatrix} ik \\ s \end{Bmatrix} U_{\substack{h\\ s}} \equiv \text{RICCI's ``second covariant derivative,''}$$

$$R_{\substack{h\\ ik}} \equiv R_{h/ik} \equiv \frac{\partial f}{\partial V_{\substack{h\\ ik}}},$$

(6) $$W_{hh} \equiv \Delta U_h \equiv \sum_{i=1}^{n}\sum_{k=1}^{n} A_{ik} U_{\substack{h\\ i}} U_{\substack{h\\ k}} \equiv \text{BELTRAMI's first differential parameter},$$

(7) $$W_{hl} \equiv \nabla U_h U_l \equiv \sum_{i=1}^{n}\sum_{k=1}^{n} A_{ik} U_{\substack{h\\ i}} U_{\substack{l\\ k}} \equiv \text{BELTRAMI's mixed differential parameter},$$

$$T_{hl} \equiv \frac{\partial f}{\partial W_{hl}},$$

$$\underset{l_1 \dots l_\mu}{x_i} \equiv x_{i/l_1 \dots l_\mu} \equiv \frac{\partial^\mu x_i}{\partial z_{l_1} \cdots \partial z_{l_\mu}},$$

$$\underset{l_1 \dots l_\mu}{\rho_i} \equiv \rho_{i/l_1 \dots l_\mu} \equiv \frac{\partial f}{\partial x_{i/l_1 \dots l_\mu}},$$

(8) $$\underset{st}{y_l} \equiv y_{l/st} \equiv \underset{st}{x_l} + \sum_{i=1}^{n}\sum_{k=1}^{n} \begin{Bmatrix} ik \\ l \end{Bmatrix} x_{i/s} x_{k/t},$$

$$\underset{st}{\sigma_l} \equiv \sigma_{l/st} \equiv \frac{\partial f}{\partial (y_{l/st})},$$

(9) $$h_{ik} \equiv \sum_{r=1}^{n}\sum_{s=1}^{n} a_{rs} x_{r/i} x_{s/k}, \qquad \gamma_{ik} = \frac{\partial f}{\partial h_{ik}},$$

(10) $$h_{i,\,kl} \equiv \sum_{r=1}^{n}\sum_{s=1}^{n} a_{rs} \underset{i}{x_r} \underset{kl}{y_s}, \qquad \gamma_{i,\,kl} = \frac{\partial f}{\partial h_{i,\,kl}},$$

(11) $$h_{ik,\,lj} \equiv \sum_{r=1}^{n}\sum_{s=1}^{n} a_{rs} \underset{ik}{y_r} \underset{lj}{y_s}, \qquad \gamma_{ik,\,lj} = \frac{\partial f}{\partial h_{ik,\,lj}}.$$

We define symbols H_{ik} by the equations,

(12) $$\sum_{i=1}^{m} h_{il} H_{ik} = \epsilon_{kl} \qquad (k, l = 1, 2, \cdots m;\ \epsilon_{kl} = 0 \text{ if } k \neq l;\ \epsilon_{kk} = 1),$$

and put

(13) $$\Lambda_{ik,\,lj} \equiv h_{ik,\,lj} - \sum_{p=1}^{m}\sum_{q=1}^{m} H_{pq} h_{p,\,ik} h_{q,\,lj},$$

$$\lambda_{ik,\,lj} \equiv \frac{\partial f}{\partial \Lambda_{ik,\,lj}}.$$

(14) $$X a_{ik} = -\sum_{r=1}^{n} (a_{rk} \underset{i}{\xi_r} + a_{ir} \underset{k}{\xi_r}).^*$$

The quantities A_{ik} satisfy the relations

(15) $$\sum_{i=1}^{n} a_{il} A_{ik} = \epsilon_{kl},$$

from which we obtain by means of (14)

(16) $$X A_{ik} = +\sum_{r=1}^{n} (A_{rk} \underset{r}{\xi_i} + A_{ir} \underset{r}{\xi_k}).$$

(17) $$X \underset{l}{a_{ik}} = -\sum_{r=1}^{n} (\underset{l}{a_{rk}} \underset{i}{\xi_r} + \underset{l}{a_{ir}} \underset{k}{\xi_r} + \underset{r}{a_{ik}} \underset{l}{\xi_r} + a_{rk} \underset{il}{\xi_r} + a_{ir} \underset{kl}{\xi_r}).^*$$

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* Cf. *Invariants*, I, p. 75.

From this and the relation defining $\left[\begin{smallmatrix} ik \\ l \end{smallmatrix}\right]$ we have

$$X\begin{bmatrix} ik \\ l \end{bmatrix} = -\sum_{r=1}^{n}\left(\begin{bmatrix} rk \\ l \end{bmatrix}\underset{i}{\xi_r} + \begin{bmatrix} ir \\ l \end{bmatrix}\underset{k}{\xi_r} + \begin{bmatrix} ik \\ r \end{bmatrix}\underset{l}{\xi_r} + a_{lr}\underset{ik}{\xi_r}\right), \tag{18}$$

and

$$X\begin{Bmatrix} ik \\ l \end{Bmatrix} = -\sum_{r=1}^{n}\left(\begin{Bmatrix} rk \\ l \end{Bmatrix}\underset{i}{\xi_r} + \begin{Bmatrix} ir \\ l \end{Bmatrix}\underset{k}{\xi_r} - \begin{Bmatrix} ik \\ r \end{Bmatrix}\underset{r}{\xi_l}\right) - \underset{ik}{\xi_l}. \tag{19}$$

Since

$$X(\underset{l_1\ldots l_\mu l_{\mu+1}}{U}) = \frac{\partial(XU_{l_1\ldots l_\mu})}{\partial x_{l_{\mu+1}}} - \sum_{r=1}^{n}\frac{\partial(\underset{l_1\ldots l_\mu}{U})}{\partial x_r}\underset{l_{\mu+1}}{\xi_r},\text{*} \tag{20}$$

we have from

$$X U_h = 0, \tag{21}$$

$$X\underset{i}{U_h} = -\sum_{r=1}^{n}\underset{r}{U_h}\,\underset{i}{\xi_r}, \tag{22}$$

and

$$X\underset{ik}{U_h} = -\sum_{r=1}^{n}\left\{\underset{rk}{U_h}\,\underset{i}{\xi_r} + \underset{ir}{U_h}\,\underset{k}{\xi_r} + \underset{r}{U_h}\,\underset{ik}{\xi_r}\right\}. \tag{23}$$

From (5) and (23) we have

$$X\underset{ik}{V_h} = -\sum_{r=1}^{n}\left\{\underset{rk}{V_h}\,\underset{i}{\xi_r} + \underset{ir}{V_h}\,\underset{k}{\xi_r}\right\}. \tag{24}$$

From (7), (16), and (22) follows

$$XW_{hl} = 0. \tag{25}$$

From (2) and (20) we have

$$Y(\underset{k}{x_i}) = \sum_{r=1}^{n}\underset{k}{x_r}\,\underset{r}{\xi_i} - \sum_{r=1}^{m}\underset{r}{x_i}\,\underset{k}{\zeta_r}, \tag{26}$$

whence

$$Y(\underset{kl}{x_i}) = \sum_{r=1}^{n}\sum_{s=1}^{n}\underset{k}{x_r}\,\underset{l}{x_s}\,\underset{rs}{\xi_i} + \sum_{r=1}^{n}\underset{kl}{x_r}\,\underset{r}{\xi_i} - \sum_{r=1}^{m}(\underset{rl}{x_i}\,\underset{k}{\zeta_r} + \underset{kr}{x_i}\,\underset{l}{\zeta_r}) - \sum_{r=1}^{m}\underset{r}{x_i}\,\underset{kl}{\zeta_r}. \tag{27}$$

From (8), (19), (26) and (27) we have

$$Y(\underset{st}{y_l}) = \sum_{r=1}^{n}\underset{st}{y_r}\,\underset{r}{\xi_l} - \sum_{r=1}^{m}([\underset{rt}{y_l}\,\underset{s}{\zeta_r} + \underset{rs}{y_l}\,\underset{t}{\zeta_r}] + \underset{r}{x_l}\,\underset{st}{\zeta_r}). \tag{28}$$

From (14) and (26) we have

$$Y(h_{ik}) = -\sum_{r=1}^{m}(h_{rk}\underset{i}{\zeta_r} + h_{ir}\underset{k}{\zeta_r}). \tag{29}$$

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* LIE-ENGEL, *Transformationsgruppen*, vol. 1, p. 545.

Similarly we obtain

$$(30)\quad Y(h_{i,\,kl}) = -\sum_{r=1}^{m}\big(h_{r,\,kl}\underset{i}{\zeta_r} + h_{i,\,rl}\underset{k}{\zeta_r} + h_{i,\,kr}\underset{l}{\zeta_r}\big) - \sum_{r=1}^{m} h_{ir}\underset{kl}{\zeta_r},$$

and

$$(31)\quad Y(h_{ik,\,lj}) = -\sum_{r=1}^{m}\big(h_{rk,\,lj}\underset{i}{\zeta_r} + h_{ir,\,lj}\underset{k}{\zeta_r} + h_{ik,\,rj}\underset{l}{\zeta_r} + h_{ik,\,lr}\underset{j}{\zeta_r}\big) - \sum_{r=1}^{m}\big(h_{r,\,lj}\underset{ik}{\zeta_r} + h_{r,\,ik}\underset{lj}{\zeta_r}\big).$$

In the same way that (16) is derived from (14) and (15), we have from (12) and (29)

$$(32)\qquad Y(H_{ik}) = +\sum_{r=1}^{m}\big(H_{rk}\underset{r}{\zeta_i} + H_{ir}\underset{r}{\zeta_k}\big).$$

From (13), (30), (31), and (32) we have

$$(33)\qquad Y(\Lambda_{ik,\,lj}) = -\sum_{r=1}^{m}\big(\underset{i}{\zeta_r}\Lambda_{rk,\,lj} + \underset{k}{\zeta_r}\Lambda_{ir,\,lj} + \underset{l}{\zeta_r}\Lambda_{ik,\,rj} + \underset{j}{\zeta_r}\Lambda_{ik,\,lr}\big).$$

Two special cases of this relation are important:

$$(34)\qquad Y(\Lambda_{ii,\,ii}) = -4\sum_{r=1}^{m}\underset{i}{\zeta_r}\Lambda_{ri,\,ii},$$

$$(35)\qquad Y(\Lambda_{ik,\,ik}) = -2\sum_{r=1}^{m}\big(\underset{i}{\zeta_r}\Lambda_{rk,\,ik} + \underset{k}{\zeta_r}\Lambda_{ir,\,ik}\big).$$

The differential equations for the parameters of the first type and μth order are obtained by equating to zero the coefficients of the various derivatives of the ξ's in the expression

$$(36)\quad \sum_{i=1}^{n}\sum_{k=1}^{i}\Big\{p_{ik}Xa_{ik} + \sum_{l_1=1}^{n}\underset{l_1}{p_{ik}}X\underset{l_1}{a_{ik}} + \cdots + \sum_{l_1=1}^{n}\sum_{l_2=1}^{l_1}\cdots\sum_{l_{\mu-1}=1}^{l_{\mu-2}}\underset{l_1\ldots l_{\mu-1}}{p_{ik}}\,X\underset{l_1\ldots l_{\mu-1}}{a_{ik}}\Big\}$$
$$+\sum_{h=1}^{m}\Big\{\sum_{l_1=1}^{n}\underset{l_1}{Q_h}X(\underset{l_1}{U_h}) + \cdots + \sum_{l_1=1}^{n}\sum_{l_2=1}^{l_1}\cdots\sum_{l_\mu=1}^{l_{\mu-1}}\underset{l_1\ldots l_\mu}{Q_h}\,X(\underset{l_1\ldots l_\mu}{U_h})\Big\}.$$

The equations for the parameters of the second type are similarly formed, save that the derivatives of the x's replace those of the U's, and, if we use the variables z, the coefficients of the derivatives of the ζ's as well as those of the ξ's must be considered.

The variables $a_{ik/l}$ can in these equations be replaced by the three index symbols $\left[{ik \atop l}\right]$,* and the equations (5) and (8) of the present section show that the variables $U_{h/ik}$ and $x_{l/st}$ can be replaced by $V_{h/ik}$ and $y_{l/st}$ respectively.

The variables a_{ik} can be replaced by the A_{ik}, and in considering parameters of the second type it is convenient to make this substitution.

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* Cf. *Invariants*, I, p. 78.

The equations for parameters of the first type of order unity, and of orders not exceeding two, may be derived from the expressions

$$(37)\qquad \sum_{i=1}^{n}\sum_{k=1}^{i} p_{ik} X(a_{ik}) + \sum_{h=1}^{m}\sum_{i=1}^{n} Q_{h/i} X(U_{h/i}),$$

and

$$(38)\qquad \sum_{i=1}^{n}\sum_{k=1}^{i} p_{ik} X(a_{ik}) + \sum_{i=1}^{n}\sum_{k=1}^{i}\sum_{l=1}^{n} q_{ik/l} X\begin{bmatrix} ik \\ l \end{bmatrix} + \sum_{h=1}^{m}\sum_{i=1}^{n} Q_{h/i} X(U_{h/i}) + \sum_{h=1}^{m}\sum_{i=1}^{n}\sum_{k=1}^{i} R_{h/ik} X(V_{h/ik}),$$

respectively.

The equations for the parameters of the second type of order unity, and of orders not exceeding two, may be determined from

$$(39)\qquad \sum_{i=1}^{n}\sum_{k=1}^{i} P_{ik} Y(A_{ik}) + \sum_{h=1}^{m}\sum_{i=1}^{n} \rho_{i/h} Y(x_{i/h}),$$

and

$$(40)\qquad \sum_{i=1}^{n}\sum_{k=1}^{i} P_{ik} Y(A_{ik}) + \sum_{i=1}^{n}\sum_{k=1}^{i}\sum_{l=1}^{n} q_{ik/l} Y\begin{bmatrix} ik \\ l \end{bmatrix} + \sum_{h=1}^{m}\sum_{i=1}^{n} \rho_{i/h} Y(x_{i/h}) + \sum_{h=1}^{n}\sum_{i=1}^{m}\sum_{k=1}^{i} \sigma_{h/ik} Y(y_{h/ik}),$$

respectively.

We shall find it convenient to denote any equation by the $\xi_{r/i}$, $\xi_{r/ik}$, $\zeta_{r/i}$ or $\zeta_{r/ik}$ of which its left-hand member is, in the above expressions, the coefficient.

If we proceed to form the equations, we notice that the introduction of the variables $V_{h/ik}$ and $y_{h/ik}$ has removed the second derivatives of the ξ's from the expressions (38) and (40), save as they appear in the expressions $X\left[{ik \atop l}\right]$.

The corresponding equations are, then,

$$(41)\qquad (\xi_{r/ik})\qquad -\sum_{l=1}^{n} a_{lr} q_{ik/l} = 0 \quad (r, i = 1, 2, \cdots, n;\ k = 1, 2, \cdots, i).$$

But these are equivalent to *

$$(42)\qquad q_{ik/l} = 0 \qquad (i, l = 1, 2, \cdots, n;\ k = 1, 2, \cdots, i).$$

Hence in both cases the $q_{ik/l}$ and the corresponding equations $\xi_{r/ik}$ may be dismissed from consideration.

The equations then take the form:

I_1. Type I, order 1,

$$(\xi_{r/i})\qquad -\sum_{k=1}^{n}(1+\epsilon_{ik}) a_{rk} p_{ik} - \sum_{h=1}^{m} U_{h/r} Q_{h/i} = 0 \qquad \begin{pmatrix} r, i = 1, 2, \cdots n \\ \epsilon_{ik} = 0,\ i \neq k \\ \epsilon_{ii} = 1 \end{pmatrix}.$$

* Cf. *Invariants*, I, p. 78.

I_2. Type I, orders 1 and 2,

$$(\underset{i}{\xi_r}) \qquad -\sum_{k=1}^{n}(1+\epsilon_{ik})a_{rk}p_{ik}-\sum_{h=1}^{m}\sum_{k=1}^{n}(1+\epsilon_{ik})\underset{rk}{V_h}\underset{ik}{R_h}-\sum_{h=1}^{m}\underset{r}{U_h}\underset{i}{Q_h}=0$$

$$(r,\, i=1,\,2,\,\cdots,\,n).$$

II_1. Type II, order 1,

$$(\underset{r}{\xi_i}) \qquad \sum_{k=1}^{n}(1+\epsilon_{ik})A_{rk}P_{ik}+\sum_{k=1}^{m}\underset{k}{x_r}\underset{k}{\rho_i}=0 \qquad (r,\, i=1,\,2,\,\cdots,\,n),$$

$$(\underset{i}{\zeta_r}) \qquad -\sum_{k=1}^{n}\underset{r}{x_k}\underset{i}{\rho_k}=0 \qquad (r,\, i=1,\,2,\,\cdots,\,m).$$

II_2. Type II, orders 1 and 2,

$$(\underset{r}{\xi_i}) \qquad \sum_{k=1}^{n}(1+\epsilon_{ik})A_{rk}P_{ik}+\sum_{k=1}^{m}\underset{k}{x_r}\underset{k}{\rho_i}+\sum_{k=1}^{m}\sum_{l=1}^{k}\underset{kl}{y_r}\underset{kl}{\sigma_i}=0 \quad (r,\, i=1,\,2,\,\cdots,\,n),$$

$$(\underset{i}{\zeta_r}) \qquad -\left\{\sum_{k=1}^{n}\underset{r}{x_k}\underset{i}{\rho_k}+\sum_{k=1}^{n}\sum_{l=1}^{m}(1+\epsilon_{il})\underset{rl}{y_k}\underset{il}{\sigma_k}\right\}=0 \qquad (r,\, i=1,\,2,\,\cdots,\,m).$$

$$(\underset{ik}{\zeta_r}) \qquad -\sum_{l=1}^{n}\underset{r}{x_l}\underset{ik}{\sigma_l}=0 \qquad (r,\, i,\, k=1,\,2,\,\cdots,\,m;\; k\leqq i)$$

The explicit form of the equations for parameters of higher order is, for our present purpose, unessential.

§ 3. *Parameters of the first type and first order.*

THEOREM. *If* $m \leqq n$ *there are in general*

$$\frac{m(m+1)}{2}$$

parameters of the first type and first order.

If $m \geqq n$ *there are*

$$\frac{n(n+1)}{2}+n(m-n)=nm-\frac{n(n-1)}{2}$$

such parameters.

They may be taken as the Beltramian parameters

$$W_{hh}\equiv\Delta U_h\equiv\sum_{i=1}^{n}\sum_{k=1}^{n}A_{ik}\underset{i}{U_h}\underset{k}{U_h},$$

$$W_{hl}\equiv\nabla U_h U_l\equiv\sum_{i=1}^{n}\sum_{k=1}^{n}A_{ik}\underset{i}{U_h}\underset{k}{U_l}.$$

The equations for the parameters are

$$(\underset{i}{\xi_r}) \qquad -\sum_{k=1}^{n}(1+\epsilon_{ik})a_{rk}p_{ik}-\sum_{h=1}^{m}\underset{r}{U_h}\,\underset{i}{Q_h}=0 \quad (r,\, i=1,\,2,\,\cdots,\,n).$$

We will suppose the equations arranged in the following order,

$$\underset{1}{\xi_1},\ \underset{1}{\xi_2},\ \cdots,\ \underset{1}{\xi_n};\ \underset{2}{\xi_1},\ \underset{2}{\xi_2},\ \cdots,\ \underset{2}{\xi_n};\ \cdots;\ \underset{n}{\xi_1},\ \underset{n}{\xi_2},\ \cdots,\ \underset{n}{\xi_n};$$

and the variables p_{ik} and $Q_{h/i}$ in the order,

$$p_{11},\ p_{12},\ \cdots,\ p_{1n};\ p_{22},\ p_{23},\ \cdots,\ p_{2n};\ \cdots;\ p_{n-1\,n-1},\ p_{n-1\,n};\ p_{n\,n};$$

$$\underset{1}{Q_1},\ \underset{2}{Q_1},\ \cdots,\ \underset{n}{Q_1};\ \underset{1}{Q_2},\ \underset{2}{Q_2},\ \cdots,\ \underset{n}{Q_2};\ \cdots;\ \underset{1}{Q_m},\ \underset{2}{Q_m},\ \cdots,\ \underset{n}{Q_m}.$$

The rows of the matrix of the coefficients of these equations will be denoted by the symbols $\xi_{r/i}$ used to represent the corresponding equations. The columns of the matrix will be denoted by the symbols p_{ik} of which the corresponding elements are coefficients.

We suppose the variables so numbered that the determinant of order $n(n+1)/2$ formed from the rows

$$\underset{1}{\xi_1},\ \underset{1}{\xi_2},\ \cdots,\ \underset{1}{\xi_n};\ \underset{2}{\xi_2},\ \underset{2}{\xi_3},\ \cdots,\ \underset{2}{\xi_n};\ \cdots;\ \underset{n-1}{\xi_{n-1}},\ \underset{n-1}{\xi_n};\ \underset{n}{\xi_n},$$

and the columns

$$p_{11},p_{12},\cdots,p_{1n};\ p_{22},p_{23},\cdots,p_{2n};\ p_{33},p_{34},\cdots,p_{3n};\ \cdots;\ p_{n-1\,n-1},p_{n-1\,n};p_{n\,n},$$

does not vanish. This is always possible.*

Suppose now $m=1$, i. e., that a single function U_1 is considered. Build the determinants whose rows are, in order

$$\underset{1}{\xi_1},\ \underset{1}{\xi_2},\ \cdots,\ \underset{1}{\xi_n};\ \underset{2}{\xi_1},\ \underset{2}{\xi_2},\ \cdots,\ \underset{2}{\xi_n};\ \underset{3}{\xi_2},\ \underset{3}{\xi_3},\ \cdots,\ \underset{3}{\xi_n};\ \underset{4}{\xi_3},\underset{4}{\xi_4}\cdots,\ \underset{4}{\xi_n};\ \cdots;\ \underset{n}{\xi_{n-1}},\ \underset{n}{\xi_n};$$

and whose columns are, in order

$$p_{11}p_{12}\cdots p_{1n};\ \underset{2}{Q_1},p_{22},\cdots,p_{2n};\ \underset{3}{Q_1},p_{33},\cdots,p_{3n};\ \underset{4}{Q_1},p_{44},\cdots,p_{4n};\ \cdots;\ \underset{n}{Q_1},p_{n\,n}.$$

This determinant has the peculiarity that coaxial with its principal diagonal lie minors of order $n, n, n-1, n-2, \cdots, 2$, respectively, and that at the right of these minors and above them the elements are all zeros. Hence the determinant consists of the product of these minors. These minors are n in number. The first is the discriminant a of the form. The remaining $n-1$ are all polynomials in the derivatives $U_{1/k}$, among the coefficients of each one of which there is at least one which does not vanish. Hence there can be assigned values of $U_{1/1}\cdots U_{1/n}$

* Cf. *Invariants*, I, p. 88.

such that *no one* of these minors vanishes. Hence none of them vanishes identically. Consequently the determinant we have just formed does not vanish identically.

Take now $m = 2$, and form the determinant, whose rows, in order, are

$$\underset{1}{\xi_1}, \underset{1}{\xi_2}, \cdots, \underset{1}{\xi_n}; \underset{2}{\xi_1}, \underset{2}{\xi_2}, \cdots, \underset{2}{\xi_n}; \underset{3}{\xi_1}, \underset{3}{\xi_2}, \underset{3}{\xi_3}, \cdots, \underset{3}{\xi_n}; \underset{4}{\xi_2}, \underset{4}{\xi_3}, \underset{4}{\xi_4}, \cdots, \underset{4}{\xi_n}; \cdots; \underset{n}{\xi_{n-2}}, \underset{n}{\xi_{n-1}}, \underset{n}{\xi_n};$$

and whose columns, in order, are

$$p_{11}, p_{12}, \cdots, p_{1n}; \underset{2}{Q_1}, p_{22}, \cdots, p_{2n}; \underset{3}{Q_2}, \underset{3}{Q_1}, p_{33}, \cdots, p_{3n}; \underset{4}{Q_2}, \underset{4}{Q_1}, p_{44}, \cdots, p_{4n}; \cdots; \underset{n}{Q_2}, \underset{n}{Q_1}, p_{nn}.$$

To this determinant the same reasoning applies as to the one before considered. It does not, therefore, identically vanish.

The process above indicated can be continued in the same way for $m = 3, 4$, etc. For $m = 1$ the order of the determinant formed is

$$n + n + (n - 1) + (n - 2) + \cdots + 3 + 2 = \frac{n(n+1)}{2} + n - 1.$$

For $m = 2$ the order is

$$n + n + n + (n - 1) + \cdots + 4 + 3 = \frac{n(n+1)}{2} + 2n - \frac{3 \cdot 2}{2},$$

and in general the order is

$$\frac{n(n+1)}{2} + mn - \frac{m(m+1)}{2}.$$

This process can, however, be continued no further than $m = n - 1$, for then

$$\frac{n(n+1)}{2} + mn - \frac{m(m+1)}{2} = n^2,$$

and all the rows of the matrix have been used. We have therefore proved that if $m \leqq n - 1$ functions U are considered, the matrix of the equations for the parameters of the first type and order contains at least one non-vanishing determinant of order

$$\frac{n(n+1)}{2} + mn - \frac{m(m+1)}{2},$$

and if $m \geqq n - 1$ it contains at least one non-vanishing determinant of order n^2. Hence

If $m \leqq n - 1$ *there are at least*

$$\frac{n(n+1)}{2} + mn - \frac{m(m+1)}{2}$$

independent equations in the system which determines the parameters of first order and type. If $m \geqq n-1$ *all the equations of this system are independent.*

It remains now to show that if $m \leqq n-1$ there are *not more than*

$$\frac{n(n+1)}{2} + mn - \frac{m(m+1)}{2}$$

independent equations in the system. To do this we make use of the fact that certain solutions of the equations, namely, BELTRAMI's first differential parameters,

$$\Delta U_h \equiv \sum_{i=1}^{n} \sum_{k=1}^{n} A_{ik} U_{h \atop i} U_{h \atop k} \equiv W_{hh},$$

and mixed differential parameters,

$$\nabla U_h U_l \equiv \sum_{i=1}^{n} \sum_{k=1}^{n} A_{ik} U_{h \atop i} U_{l \atop k} \equiv W_{hl},$$

are known. Moreover these solutions regarded as functions of the explicitly expressed variables A_{ik}, $U_{h/l}$ are functionally independent as may be seen from the fact that if

$$U_r = x_1 + x_2 + \cdots + x_r \qquad (r \leqq m),$$

the Jacobian

$$\frac{\partial(W_{11} W_{21} W_{22} W_{31} W_{32} W_{33} \cdots W_{mm})}{\partial(A_{11} A_{21} A_{22} A_{31} A_{32} A_{33} \cdots A_{mm})}$$

has the value unity, and does not, therefore, vanish. If $m \geqq n$ the same set of values for $U_1 \cdots U_n$ shows that the Jacobian

$$\frac{\partial(W_{11} W_{21} W_{22} \cdots W_{nn}, W_{1\,n+1}, W_{2\,n+1} \cdots W_{n\,n+1} \cdots W_{1m}, \cdots, W_{nm})}{\partial(A_{11} A_{21} A_{22} \cdots A_{nn}, U_{n+1 \atop 1}, U_{n+1 \atop 2} \cdots U_{n+1 \atop n} \cdots U_{m \atop 1} \cdots U_{m \atop n})}$$

does not vanish.

We have seen that there are at least $n(n+1)/2 + mn - m(m+1)/2$ independent equations if $m \leqq n-1$ and that all the equations are independent if $m \geqq n-1$.

Hence there cannot be more than

$$\frac{n(n+1)}{2} + mn - \left\{ \frac{n(n+1)}{2} + mn - \frac{m(m+1)}{2} \right\} = \frac{m(m+1)}{2}$$

independent parameters if $m \leqq n-1$, and not more than

$$\frac{n(n+1)}{2} + mn - n^2 = \frac{n(n+1)}{2} + n(m-n)$$

if $m \geqq n-1$. But we have actually formed this number of independent parameters. The theorem is therefore proved.

§ 4. *Parameters of the first type and second order.*

THEOREM: *The number of parameters of the first type and second order is, in general,*

$$\frac{mn(m+1)}{2}-\frac{(n-m)(n-m-1)}{2}, \qquad \textit{if } m \leqq n,$$

and

$$\frac{mn(n+1)}{2}, \qquad \textit{if } m \geqq n.$$

The equations determining the parameters of order not exceeding two may be written:

$$(\underset{i}{\xi_r}) \qquad -\left\{\sum_{k=1}^{n}(1+\epsilon_{ik})a_{rk}p_{ik}+\sum_{h=1}^{m}\sum_{k=1}^{n}(1+\epsilon_{ik})\underset{rk}{V_h}\underset{ik}{R_h}+\sum_{h=1}^{m}\underset{r}{U_h}\underset{i}{Q_h}\right\}=0$$

$$(r,\ i=1,2,\cdots,n).$$

If we omit the terms

$$\sum_{h=1}^{m}\underset{r}{U_h}\underset{i}{Q_h},$$

there remain the equations which determine the simultaneous invariants of order zero for the $m+1$ forms

$$\sum_{i=1}^{n}\sum_{k=1}^{n}a_{ik}dx_i dx_k,$$

$$\phi_h=\sum_{i=1}^{n}\sum_{k=1}^{n}\underset{ik}{V_h}dx_i dx_k \qquad (h=1,2,\cdots,m).$$

But these equations are all independent. *

Consequently the equations of the present case, which differ from these only by the addition of terms are also all independent.

The number of equations is n^2, the number of variables,

$$\frac{mn(n+1)}{2}+\frac{n(n+1)}{2}+mn.$$

The number of parameters of orders one and two is, therefore,

$$\frac{mn(n+1)}{2}+\frac{n(n+1)}{2}+mn-n^2=\frac{mn(n+3)}{2}-\frac{n(n-1)}{2}.$$

The number of parameters of the first order is

$$\frac{m(m+1)}{2}, \qquad \textit{if } m \leqq n-1,$$

and

$$\frac{n(n+1)}{2}+mn-n^2, \qquad \textit{if } m \geqq n-1.$$

---

* This theorem is involved in the theorem concerning simultaneous invariants, (*Invariants*, I, p. 91). The proof follows the same lines as that used in *Invariants* I, § 5, for a similar purpose.

Hence the number of parameters of the second order is

$$\frac{mn(n+1)}{2}-\frac{(n-m)(n-m-1)}{2}, \qquad \textit{if } m \leqq n-1;$$

and

$$\frac{mn(n+1)}{2}, \qquad \textit{if } m \geqq n-1.*$$

Q. E. D.

§ 5. *Parameters of the second type and first order.*

THEOREM. *There are in general no parameters of the second type and first order.*

The equations for the parameters of the second type and first order may be written:

$$(\underset{r}{\xi_i}) \qquad \sum_{k=1}^{n}(1+\epsilon_{ik})A_{rk}P_{ik}+\sum_{k=1}^{m}\underset{k}{x_r}\underset{k}{\rho_i}=0 \qquad (r, i=1, 2, \cdots, n),$$

$$(\underset{i}{\zeta_r}) \qquad -\sum_{k=1}^{n}\underset{r}{x_k}\underset{i}{\rho_k}=0 \qquad (r, i=1, 2, \cdots, m).$$

We note that the equation $\xi_{i/r}$ of this set involves the indices i and r in the in the same way as does the equation $\xi_{r/i}$ of the set which determines the parameters of the *first* type and order. Consequently we shall expect to find that the functions

$$h_{ik} \equiv \sum_{r=1}^{n}\sum_{s=1}^{n}a_{rs}\underset{i}{x_r}\underset{k}{x_s},$$

are solutions of these equations; and this expectation is confirmed by equations (29) of § 2.

Moreover, the operations by which we proved the independence of equations $\xi_{r/i}$ of § 3 can be repeated step for step here. Hence as $m \leqq n-1$ the equations $\xi_{r/i}$ have precisely $m(m+1)/2$ independent solutions, viz., the functions h_{ik}.

We may therefore introduce these as new variables into the equations $\zeta_{r/i}$. These, then become, however, by means of equation (29), § 2,

$$(\underset{i}{\zeta_r}) \qquad \sum_{k=1}^{m}(1+\epsilon_{ik})h_{rk}\gamma_{ik}=0,$$

i. e., the equations which determine the invariants of order zero for a form in m variables. This system consists of m^2 equations in $m(m+1)/2$ variables. Its matrix contains at least one determinant of order $m(m+1)/2$ which does not

* Evidently

$$\frac{mn(n+1)}{2}=\frac{mn(n+1)}{2}-\frac{(n-m)(n-m-1)}{2} \qquad \text{when} \quad m=n-1 \quad \text{or} \quad m=n.$$

formally vanish identically.* Now the h_{ik}'s are independent rational integral functions of the $(n+1)/2 + mn > m(m+1)/2$ variables a_{ik}, $x_{i/k}$. Hence these variables can be given such values that this determinant does not vanish. This set of m^2 equations contains therefore in general as many independent equations as variables. Hence it has no solutions.

Hence *there are no parameters of the second type and first order.*

§ 6. *Parameters of the second type and second order.*

Theorem. *The number of parameters of the second type and second order is, in general,*

$$\frac{m(m+3)(m^2-m+2)}{8}, \qquad \text{if} \qquad \frac{m(m+3)}{2} \leqq n,$$

and

$$\frac{(n-m)m(m+3)}{2} - n\frac{(n-1)}{2}, \qquad \text{if} \qquad \frac{m(m+3)}{2} \geqq n,$$

The equations which the parameters of the second order and type must satisfy are:

$$(\underset{r}{\xi_i}) \qquad \sum_{k=1}^{n}(1+\epsilon_{ik})A_{rk}P_{ik} + \sum_{k=1}^{m}\underset{k}{x_r}\underset{k}{\rho_i} + \sum_{k=1}^{m}\sum_{l=1}^{k}\underset{kl}{y_r}\underset{kl}{\sigma_i} = 0 \qquad (r, i = 1, 2, \cdots n),$$

$$(\underset{i}{\zeta_r}) \qquad -\sum_{k=1}^{n}x_k\underset{r}{\rho}_{\underset{i}{k}} - \sum_{k=1}^{n}\sum_{l=1}^{m}(1+\epsilon_{il})\underset{rl}{y_k}\underset{il}{\sigma_k} = 0 \qquad (r, i = 1, 2, \cdots m),$$

$$(\underset{ik}{\zeta_r}) \qquad -\sum_{l=1}^{n}x_l\underset{ik}{\sigma_l} = 0 \qquad (r, i, k = 1, 2, \cdots m;\ k \leqq i).$$

Equations (29), (30), (31) and (33) of § 2 show that the functions h_{ik}, $h_{i,kl}$, $h_{ik,lj}$ and $\Lambda_{ik,lj}$ † are solutions of the equations $\xi_{i/r}$, and the results of § 3 show that the number of independent solutions of these equations is

$$\frac{m(m+1)(m+2)(m+3)}{8} \qquad \text{if} \qquad \frac{m(m+3)}{2} \leqq n,$$

and

$$\frac{mn(m+3)}{2} - \frac{n(n-1)}{2}, \qquad \text{if} \qquad \frac{m(m+3)}{2} \geqq n.$$

We shall first show that these solutions may be selected from the solutions h_{ik}, $h_{i,kl}$, $\Lambda_{ik,lj}$. To this end we arrange these functions in the following order:

---

* Cf. *Invariants*, I, p. 88.

† The functions $h_{ik,lj}$ are those which naturally present themselves first in the solution. The functions $\Lambda_{ik,lj}$ are introduced to meet the exigencies of the present section.

$$\begin{array}{llllllllll}
h_{11} & h_{12} & \cdots h_{1m} & h_{22} & \cdots h_{2m} & \cdots h_{m-1\,m-1} & h_{m-1\,m} & h_{mm} & ; \\
h_{1,11} & h_{1,12} & \cdots h_{1,1m} & h_{1,22} & \cdots h_{1,2m} & \cdots h_{1,m-1\,m-1} & h_{1,m-1\,m} & h_{1,mm} & ; \\
h_{2,11} & h_{2,12} & \cdots h_{2,1m} & h_{2,22} & \cdots h_{2,2m} & \cdots h_{2,m-1\,m-1} & h_{2,m-1\,m} & h_{2,mm} & ; \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\
h_{m,11} & h_{m,12} & \cdots h_{m,1m} & h_{m,22} & \cdots h_{m,2m} & \cdots h_{m,m-1\,m-1} & h_{m,m-1\,m} & h_{m,mm} & ;
\end{array}$$

$$\begin{array}{lllll}
\Lambda_{11,11} & & & & \\
\Lambda_{12,12} & \Lambda_{11,12} & & & \\
\Lambda_{13,13} & \Lambda_{12,13} & \Lambda_{11,13} & & \\
\Lambda_{14,14} & \Lambda_{13,14} & \Lambda_{12,14} & \Lambda_{11,14} & \\
\cdot & \cdot & \cdot & \cdot & \\
\Lambda_{1m-1,1m-1} & \Lambda_{1m-2,m-1} & \Lambda_{1m-3,1m-1} & \Lambda_{1m-4,1m-1} & \cdots \\
\Lambda_{1m,1m} & \Lambda_{1m-1,1m} & \Lambda_{1m-2,1m} & \Lambda_{1m-3,1m} & \cdots \\
\Lambda_{22,22} & \Lambda_{1m,22} & \Lambda_{1m-1,22} & \Lambda_{1m-2,22} & \cdots \\
\Lambda_{23,23} & \Lambda_{22,23} & \Lambda_{1m,23} & \Lambda_{1m-1,23} & \cdots \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\Lambda_{mm,mm} & \Lambda_{m-1\,m,mm} & \Lambda_{m-1\,m-1,mm} & \Lambda_{m-2\,m,mm} & \cdots \Lambda_{11\,mm}
\end{array}$$

The order in which the h_{ik} and $h_{i,kl}$ are arranged is obvious. The order of the Λ's is as follows: The pairs of indices $11 \cdots mm$ are supposed to be arranged in the same order as that given by the h's. The first column of Λ's consists of those in which the second pair of indices is the same as the first pair and the Λ's are arranged in the above given order of indices. Each succeeding column is obtained from the preceding by dropping the uppermost element, retaining in the remainder the second pair of indices, but replacing the first pair by the corresponding pair taken from the element next above.

Suppose now $m(m+3)/2 > n$. Form the above table and from it take all the h_{ik} and $h_{i.kl}$, but only the first $n-m$ columns of the Λ's. The number of the functions taken is

$$\frac{m(m+1)^2}{2} + \left(\frac{m(m+1)}{2} + \frac{m(m+1)}{2} - 1 + \frac{m(m+1)}{2} - 2 \right.$$
$$\left. \cdots + \frac{m(m+1)}{2} - n + m + 1\right) = \frac{nm(m+3)}{2} - \frac{n(n-1)}{2},$$

which is the number of independent solutions of the equations $\xi_{i/r}$. We must now show that these particular solutions are all independent. For this purpose we may suppose the functions just chosen arranged in the single row obtained by annexing each row of the above scheme to its predecessor.

We will then form the Jacobian, J, of these functions with respect to the variables

$$
\begin{array}{ccccccccccc}
\underset{1}{x_1} & \underset{1}{x_2} & \cdots & \underset{1}{x_m} & \underset{2}{x_2} & \cdots & \underset{2}{x_m} & \cdots & \underset{m-1}{x_{m-1}} & \underset{m-1}{x_m} & \underset{m}{x_m} \\
\underset{11}{y_1} & \underset{12}{y_1} & \cdots & \underset{1m}{y_1} & \underset{22}{y_1} & \cdots & \underset{2m}{y_1} & \cdots & \underset{m-1\,m-1}{y_1} & \underset{m-1\,m}{y_1} & \underset{mm}{y_1} \\
\underset{11}{y_2} & \underset{12}{y_2} & \cdots & \underset{1m}{y_2} & \underset{22}{y_2} & \cdots & \underset{2m}{y_2} & \cdots & \underset{m-1\,m-1}{y_2} & \underset{m-1\,m}{y_2} & \underset{mm}{y_2} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\underset{11}{y_m} & \underset{12}{y_m} & \cdots & \underset{1m}{y_m} & \underset{22}{y_m} & \cdots & \underset{2m}{y_m} & \cdots & \underset{m-1\,m-1}{y_m} & \underset{m-1\,m}{y_m} & \underset{mm}{y_m} \\
\underset{11}{y_{m+1}} & & & & & & & & & & \\
\underset{12}{y_{m+1}} & \underset{12}{y_{m+2}} & & & & & & & & & \\
\underset{13}{y_{m+1}} & \underset{13}{y_{m+2}} & & \underset{13}{y_{m+3}} & & & & & & & \\
\cdot & \cdot & \cdot & \cdot & \cdot & & & & & & \\
\underset{1,\,m-1}{y_{m+1}} & \underset{1,\,m-1}{y_{m+2}} & & \underset{1,\,m-1}{y_{m+3}} & & \cdots & & & \underset{1,\,m-1}{y_n} & & \\
\underset{1m}{y_{m+1}} & \underset{1m}{y_{m+2}} & & \underset{1m}{y_{m+3}} & & \cdots & & & \underset{1m}{y_n} & & \\
\underset{22}{y_{m+1}} & \underset{22}{y_{m+2}} & & \underset{22}{y_{m+3}} & & \cdots & & & \underset{22}{y_n} & & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & \\
\underset{mm}{y_{m+1}} & \underset{mm}{y_{m+2}} & & \underset{mm}{y_{m+3}} & & \cdots & & & \underset{mm}{y_n} & &
\end{array}
$$

and show that this Jacobian does not identically vanish. To this end we show that, for a particular form ϕ, and a particular set of relations between the x's and z's, J takes a value different from zero.

We will suppose, first, that ϕ has *constant* coefficients and that none of the minors of its discriminant of any order are zero. The relations between the x's and the z's shall be:

$$
x_r = \sum_{i=1}^{m} A_{ri} z_i + \tfrac{1}{2} \sum_{i=1}^{m} \sum_{k=1}^{m} \underset{ik}{B_r} z_i z_k \qquad (r = 1, 2, \cdots, n;\ \underset{ik}{B_r} = \underset{ki}{B_r}),
$$

where the A_{rs} have the significance we have already attached to these symbols and the $B_{r/ik}$ are constants which we shall later subject to conditions.

Since ϕ has constant coefficients,

$$
\underset{ik}{x_r} = \underset{ik}{y_r}.
$$

From the above we have

$$
\underset{i}{x_r} = A_{ri} + \sum_{k=1}^{m} \underset{ik}{B_r} z_k,
$$

and

$$
\underset{ik}{y_r} = \underset{ik}{x_r} = \underset{ik}{B_r},
$$

If $z_1 = z_2 \cdots = z_m = 0$, $x_1 = x_2 \cdots = x_n = 0$, and $\underset{i}{x_r} = A_{ri}$.

We shall consider the value of J for this set of values of the variables.

Now

$$\frac{\partial h_{ik}}{\partial x_{\underset{l}{r}}} = 0, \text{ unless } l = i \text{ or } l = k;$$

$$\frac{\partial h_{ik}}{\partial x_{\underset{i}{r}}} = \sum_{s=1}^{n} a_{rs} x_{\underset{k}{s}} \text{ if } k \neq i; \qquad \frac{\partial h_{ii}}{\partial x_{\underset{i}{r}}} = 2\sum_{s=1}^{n} a_{rs} x_{\underset{i}{s}}.$$

But in the special case in question

$$\sum_{s=1}^{n} a_{rs} x_{\underset{k}{s}} = \sum_{s=1}^{n} a_{rs} A_{sk} = \epsilon_{rk}.$$

Hence

$$\frac{\partial h_{ik}}{\partial x_{\underset{i}{r}}} = 0, \text{ if } r \neq k; \qquad \frac{\partial h_{ik}}{\partial x_{\underset{i}{k}}} = 1, \text{ if } i \neq k; \qquad \frac{\partial h_{ii}}{\partial x_{\underset{i}{i}}} = 2.$$

Now h_{ik} does not contain the variables $y_{r/st}$. Hence J breaks up into the product of two Jacobians, one J_1, that of the h_{ik} with respect to the variables $x_{r/i}$, the other J_2, that of the remaining functions with respect to the remaining variables.

It is easily seen that J_1 has the value 2^m.

Consider now J_2.

$$\frac{\partial h_{i,\,kl}}{\partial y_{\underset{st}{r}}} = 0 \text{ unless } \left\{\begin{matrix} s = k \\ t = l \end{matrix}\right\}, \qquad \text{or} \left\{\begin{matrix} t = k \\ s = l \end{matrix}\right\}, \qquad \frac{\partial h_{i,\,kl}}{\partial y_{\underset{kl}{r}}} = \sum_{s=1}^{n} a_{rs} x_{\underset{i}{s}}.$$

In the special case under consideration $\partial h_{i,\,kl}/\partial y_{r|kl} = \epsilon_{ir}$.

Now the index i in $h_{i,\,kl}$ is always $\leqq m$. Hence if $r = m+1, m+2, \cdots$ or n,

$$\frac{\partial h_{i,\,kl}}{\partial y_{\underset{st}{r}}} = 0.$$

Hence J_2 in turn breaks up into the product of two Jacobians J_2' and J_3; where

$$J_2' = \frac{\partial(h_{1,11} h_{1,12} \cdots h_{1,mm} h_{2,11} \cdots h_{2,mm} \cdots h_{m,mm})}{\partial(y_{1/11} y_{1/12} \cdots y_{1/mm} y_{2/11} \cdots y_{2/mm} \cdots y_{m/mm})}.$$

Therefore

$$J_2' \big|_{z=0} = 1.$$

The Jacobian J_3 is that of the remaining functions with respect to the remaining variables. We suppose both functions and variables to be arranged in the order given above.

Now,

$$\frac{\partial \Lambda_{ik,\,lj}}{\partial y_{r|st}} = 0, \quad \text{unless} \quad \left\{\begin{matrix} s = i \\ t = k \end{matrix}\right\}, \left\{\begin{matrix} s = k \\ t = i \end{matrix}\right\}, \left\{\begin{matrix} s = l \\ t = j \end{matrix}\right\} \text{ or } \left\{\begin{matrix} s = j \\ t = l \end{matrix}\right\},$$

But if the pairs of indices ik and lj are different,

$$\frac{\partial \Lambda_{ik,\,lj}}{\partial y_{r|ik}} = \sum_{s=1}^{n} a_{rs} \underset{lj}{y_s} - \sum_{q=1}^{m} h_{q,\,lj} \sum_{p=1}^{m} H_{pq} \sum_{s=1}^{n} a_{sr} \underset{q}{x_s},$$

and if they are the same

$$\frac{\partial \Lambda_{ik,\,ik}}{\partial y_{r|ik}} = 2 \sum_{s=1}^{n} a_{rs} \underset{ik}{y_s} - 2 \sum_{q=1}^{m} h_{q,\,ik} \sum_{p=1}^{m} H_{pq} \sum_{s=1}^{n} a_{sr} \underset{q}{x_s}.$$

In the present case, since $q \leqq m$ and $r > m$, $q \neq r$, and

$$\sum_{s=1}^{n} a_{sr} \underset{q}{x_s} = \sum_{s=1}^{n} a_{sr} A_{sq} = \epsilon_{qr} = 0\,.$$

Hence

$$\frac{\partial \Lambda_{ik,\,lj}}{\partial y_{r/ik}} = \sum_{s=1}^{n} a_{rs} \underset{lj}{B_s}, \qquad \frac{\partial \Lambda_{ik,\,ik}}{\partial y_{r/ik}} = 2 \sum_{s=1}^{n} a_{rs} \underset{ik}{B_s}.$$

From these equations we see that the only elements of J_3 arising from $\Lambda_{ik,\,lj}$ lie in the rows containing derivatives with respect to $y_{r/ik}$ and $y_{r/lj}$. Moreover in the row $y_{r|ik}$ the indices of the B_s coming from $\Lambda_{ik,\,lj}$ are l, j; and in the row $y_{r/lj}$ they are i, k. If, bearing this in mind, we consider the order in which both functions and variables are arranged, we see that the determinant J_3 contains a series of minors arranged in order with their principal diagonals lying along that of J_3, and that below and at the left of these minors all the elements of J_3 are zero. Hence J_3 is the product of these minors. The first $n-m$ of these minors are determinants of orders $1, 2, \cdots, n-m$ respectively. The remainder are of order $n-m$.

Consider a typical one of these minors, e. g., the one whose columns are $1k, 1k$; $1k-1, 1k$; $1k-2, 1k$; $\cdots 1k-(n-m-1), 1k$; and whose rows are $\underset{1k}{m+1}, \underset{1k}{m+2}, \underset{1k}{m+3}, \cdots \underset{1k}{n}$. It is

$$\begin{array}{c|ccccc|}
 & 1k,\,1k & 1k-1,\,1k & 1k-2,\,1k & \cdots & 1k-(n-m-1),\,1k \\
\hline
\underset{1k}{m+1} & 2\sum_{s=1}^{n} a_{m+1\,s} \underset{1k}{B_s} & \sum_{s=1}^{n} a_{m+1\,s} \underset{1k-1}{B_s} & \sum_{s=1}^{n} a_{m+1\,s} \underset{1k-2}{B_s} & \cdots & \sum_{s=1}^{n} a_{m+1\,s} \underset{1k-n+m+1}{B_s} \\
\underset{1k}{m+2} & 2\sum_{s=1}^{n} a_{m+2\,s} \underset{1k}{B_s} & \sum a_{m+2\,s} \underset{1k-1}{B_s} & \sum_{s=1}^{n} a_{m+2\,s} \underset{1k-2}{B_s} & \cdots & \sum_{s=1}^{n} a_{m+2\,s} \underset{1k-n+m+1}{B_s} \\
\underset{1k}{m+3} & 2\sum_{s=1}^{n} a_{m+3\,s} \underset{1k}{B_s} & \sum a_{m+3\,s} \underset{1k-1}{B_s} & \sum_{s=1}^{n} a_{m+3\,s} \underset{1k-2}{B_s} & \cdots & \sum_{s=1}^{n} a_{m+3\,s} \underset{1k-n+m+1}{B_s} \\
 & \cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot \\
\underset{1k}{n} & 2\sum_{=1}^{n} a_{ns} \underset{1k}{B_s} & \sum a_{ns} \underset{1k-1}{B_s} & \sum_{s=1}^{n} a_{ns} \underset{1k-2}{B_s} & \cdots & \sum_{s=1}^{n} a_{ns} \underset{1k-n+m+1}{B_s}
\end{array}$$

The arrangement of the functions Λ is such that the first column of this determinant contains B's which have not before appeared. The remaining columns have however all appeared. We will suppose that not all the determinants of order $n-m-1$ formed from these columns vanish. Then because of the non-vanishing of the minors of the discriminant of ϕ it is possible to choose the quantities $B_{1/1k} B_{2/1k} \cdots B_{n/1k}$ in such a way that the elements in the first column of the determinant in question shall have any values whatever. Hence they may be so chosen that the determinant does not vanish.

Beginning now with the minor of order unity in the column $11, 11$ and row $m+1$, we may choose the quantities $B_{s/11}$ so that it does not vanish. Passing to the minor of order two in the next two columns and rows, we may choose the $B_{s/12}$ so that it does not vanish and so on.

It is therefore possible to choose the B's so that J_3 does not vanish. We have found that with our choice of the A's, J_1 and J_2' did not vanish. Hence for this form ϕ and these relations between the x's and z's J does not vanish for the values $z_1 = z_2 \cdots = z_m = 0$.

Hence J does not vanish identically, and therefore the functions chosen are *independent* solutions of the equations $\xi_{i/r}$.

In the case $m(m+3)/2 \leqq n$, the solutions are all independent and the foregoing work is unnecessary.

Introduce now these solutions as new variables into the equations $\zeta_{r/i}$ and $\zeta_{r/ik}$. The latter equations become

$$\sum_{i=1}^{m} h_{ir}\, \gamma_{i,\,kl} = 0 \qquad (r,\, k,\, l = 1,\, 2,\, \cdots,\, m\,;\; l \leqq k).$$

Now the determinant $h \equiv | h_{11} \cdots h_{mm} |$ is easily seen to have the value $| A_{11} \cdots A_{mm} |$ in the case we are considering, and because of the non-vanishing of the minors of the discriminant of ϕ it follows that in this case

$$h \neq 0.$$

Hence *

$$\gamma_{i,\,kl} = 0.$$

Consequently we may dismiss the functions $h_{i,\,kl}$ and the equations $\zeta_{i/kl}$ from further consideration.

There remain only the equations $\zeta_{r/i}$ to be considered. They are m^2 in number. The number of variables is the number of independent solutions of the equations $\xi_{i/r}$ diminished by $m^2(m+1)/2$, the number of functions $h_{i,\,kl}$.

The excess of variables over equations is therefore

$$\frac{m(m+1)(m+2)(m+3)}{8} - \frac{m^2(m+1)}{2} - m^2 = \frac{m(m+3)}{8}(m^2 - m + 2),$$

$$\text{if } \frac{m(m+3)}{2} \leqq n;$$

* Cf. *Invariants*, I, p. 78.

and

$$\phi(n, m) \equiv (n - m)\frac{m(m+3)}{2} - \frac{n(n-1)}{2}, \qquad \text{if } \frac{m(m+3)}{2} \geqq n.$$

It is important to note some properties of the function $\phi(n, m)$.

$$\frac{\partial \phi(n, m)}{\partial n} = \frac{m(m+3)}{2} - (n - \tfrac{1}{2}).$$

Hence if $m(m+3)/2 \geqq n$, in which case only is $\phi(n, m)$ used, $\partial\phi/\partial n > 0$. But the least value of n is $m + 1$, and $\phi(m+1, m) = m$. Hence if $m + 1 \leqq n \leqq m(m+3)/2$, then

$$\phi(n, m) > m.$$

The number of variables is

$$\phi(n, m) + m^2 > m^2 + m.$$

Of these $m(m+1)/2$ are the functions $h_{i,k}$, hence of the variables at least

$$\phi(nm) + m^2 - \frac{m(m+1)}{2} > \frac{m(m+1)}{2}$$

are among the functions $\Lambda_{ik,\,lj}$.

Consider now that part of the new equations $\zeta_{r/i}$ which arises from the functions h_{ik} and $\Lambda_{ik,\,ik}$. We have seen that these latter functions may always be included among the new variables. Let us now put

$$\underset{ik}{B_1} = \underset{ik}{B_2} = \cdots = \underset{ik}{B_n} = B_{ik} \qquad (i, k = 1, 2, \cdots m).$$

Then

$$\Lambda_{ik,\,lj} = B_{ik} B_{lj} \left\{ \sum_{r=1}^{n} \sum_{s=1}^{n} a_{rs} - \sum_{r=1}^{m} \sum_{s=1}^{m} H_{rs} \right\}.$$

The expression in the brace is common to all the Λ's. It does not identically vanish, for in the present special case the first sum involves a_{nn} while the second does not. The a's may therefore be so taken that the expression does not vanish. Denote its value by M. Then

$$\Lambda_{ik,\,lj} = M B_{ik} B_{lj},$$

but

$$Y(\Lambda_{ii,\,ii}) = -4 \sum_{r=1}^{m} \underset{i}{\zeta_r} \Lambda_{ri,\,ii},$$

and

$$Y(\Lambda_{ik,\,ik}) = -2 \sum_{r=1}^{m} (\underset{i}{\zeta_r} \Lambda_{rk,\,ik} + \underset{k}{\zeta_r} \Lambda_{ir,\,ik}).$$

Hence, in the present case,

$$Y(\Lambda_{ii,\,ii}) = -4M\sum_{r=1}^{m} \underset{i}{\zeta_r} B_{ri} B_{ii},$$

$$Y(\Lambda_{ik,\,ik}) = -2M\sum_{r=1}^{m} (\underset{i}{\zeta_r} B_{rk} B_{ik} + \underset{k}{\zeta_r} B_{ir} B_{ik}).$$

Similarly

$$Y(h_{ii}) = -2\sum_{r=1}^{m} \underset{i}{\zeta_r} h_{ri},$$

and

$$Y(h_{ik}) = -\sum_{r=1}^{m} (\underset{i}{\zeta_r} h_{rk} + \underset{k}{\zeta_r} h_{ri}),$$

which in this case become

$$Y(h_{ii}) = -2\sum_{r=1}^{m} \underset{i}{\zeta_r} A_{ri},$$

$$Y(h_{ik}) = -\sum_{r=1}^{m} (\underset{i}{\zeta_r} A_{rk} + \underset{k}{\zeta_r} A_{ir}).$$

If we now form that part of the matrix of the coefficients of the equations $\zeta_{r/i}$ which arises from the functions h_{ik} and $\Lambda_{ik,\,ik}$ we see that, first, from any column, such as that of the coefficients of $\partial f/\partial\Lambda_{ik,\,ik}$ we may divide out the common factor $2MB_{ik}$, and second, after this division the form of this part of the matrix is that of the matrix of the equations for the simultaneous invariants of order zero for the two forms

$$\phi_1 = \sum_{r=1}^{m}\sum_{s=1}^{m} A_{rs}\,dx_r\,dx_s,$$

$$\phi_2 = \sum_{r=1}^{m}\sum_{s=1}^{m} B_{rs}\,dx_r\,dx_s.$$

But we have seen* that it is then possible so to take the B_{rs} that there shall be a non-vanishing determinant of order m^2 in this matrix.

It therefore follows that the matrix of the equations of the present problem contains at least one determinant of order m^2 which does not in this special case vanish. Hence it does not *identically* vanish. *Hence the equations* $\zeta_{r/i}$ *are all independent.*

As the equations are all independent the *number of parameters is the excess of variables over equations* and the theorem is proved.

It may be noted that the choice of the quantities $B_{r/ik}$ by means of which the independence of the equations $\zeta_{r/i}$ is proved, is such as to make the Jacobian J vanish. This does not however invalidate the reasoning. For let K be the non-vanishing determinant of the matrix; J and K are both polynomials in the vari-

* Cf. note (§ 4), also *Invariants*, I, p. 91.

ables $y_{r/ik}$, and neither vanishes identically, for we have seen that it is possible to make either of them take a value different from zero. Hence it is possible to find values of the $y_{r,\,ik}$ such that they do not vanish simultaneously.

§ 7. *Parameters of order greater than* 2.

In the case of parameters of the first type, and also in that of parameters of the second type, provided the first method of determining the equations is used, the equations for the parameters of order μ differ from those for the invariants of order $\mu - 1$ only by the addition of terms. Consequently if the equations for the invariants of order $\mu - 1$ are independent, so are those for the parameters of order μ. But if $n > 2$ the equations for the invariants of order $\mu \geqq 2$ are all independent. Hence the equations for the parameters of order $\mu \geqq 3$ are all independent. Hence:

The number of parameters, of either type, of order not exceeding μ, is, if $\mu \geqq 3$ equal to the excess of the number of variables over the number of equations determining those parameters. The cases $\mu = 1, 2$ have been treated separately.

Suppose now $\mu \geqq 3$. The equations are all independent. The number of equations is, for parameters of either type, equal to the number of derivatives of the ξ's of orders not exceeding μ. The variables in the equations are of two kinds; first the a_{ik}'s and their derivatives of orders not exceeding $\mu - 1$; and, second, for parameters of the first type, the derivatives of the m functions U with respect to the n variables x; or, for parameters of the second type, the derivatives of the $n - m$ dependent, with respect to the m independent x's; the derivatives in both cases being of all orders not exceeding μ.

Hence the number of equations is, in both cases,

$$n\left\{\frac{(n+\mu)!}{n!\,\mu!} - 1\right\}.$$

The number of variables is, for parameters of the first type,

$$\frac{n(n+1)}{2}\,\frac{(n+\mu-1)!}{n!(\mu-1)!} + m\left\{\frac{(n+\mu)!}{n!\,\mu!} - 1\right\},$$

and, for parameters of the second type,

$$\frac{n(n+1)}{2}\,\frac{(n+\mu-1)!}{n!(\mu-1)!} + (n-m)\left\{\frac{(m+\mu)!}{m!\,\mu!} - 1\right\}.$$

The numbers of solutions are, then,

$$\frac{n(n+1)}{2}\,\frac{(n+\mu-1)!}{n!(\mu-1)!} - n\left\{\frac{(n+\mu)!}{n!\,\mu!} - 1\right\} + m\left\{\frac{(n+\mu)!}{n!\,\mu!} - 1\right\},$$

and

$$\frac{n(n+1)}{2}\frac{(n+\mu-1)!}{n!(\mu-1)!}-n\left\{\frac{(n+\mu)!}{n!\mu!}-1\right\}+(n-m)\left\{\frac{(m+\mu)!}{m!\mu!}-1\right\},$$

in the two cases respectively.

But of these solutions,

$$\frac{n(n+1)}{2}\frac{(n+\mu-1)!}{n!(\mu-1)!}-n\left\{\frac{(n+\mu)!}{n!\mu!}-1\right\}$$

are *invariants* of order $\leqq \mu-1$.

Hence the number of *parameters* of order $\leqq \mu$ and of the first type, is

$$m\left\{\frac{(n+\mu)!}{n!\mu!}-1\right\},$$

and of the second type

$$(n-m)\left\{\frac{(m+\mu)!}{m!\mu!}-1\right\}.$$

If $\mu \geqq 4$ then the corresponding relations hold for $\mu-1$.

Hence the number of parameters of order μ of the first type is

$$\frac{m(n+\mu-1)!}{(n-1)!\,\mu!},$$

and of the second type

$$\frac{(n-m)(m+\mu-1)!}{(m-1)!\,\mu!}.$$

Here $\mu \geqq 4$, $n \geqq 3$.

The number of parameters of orders not exceeding 3 and of the first type is

$$m\left\{\frac{(n+1)(n+2)(n+3)}{6}-1\right\}=\frac{mn(n+1)(n+2)}{6}+\frac{mn(n+3)}{2},$$

and of the second type

$$(n-m)\left\{\frac{(m+1)(m+2)(m+3)}{6}-1\right\}=\frac{(n-m)m(m+1)(m+2)}{6}$$
$$+(n-m)\frac{m(m+3)}{2}.$$

The number of parameters of orders not exceeding 2 and of the first type is

$$\frac{mn(n+3)}{2}-\frac{n(n-1)}{2},$$

and of the second type

$$\frac{m(m+3)(m^2-m+2)}{8}, \qquad \text{if } \frac{m(m+3)}{2} \leqq n,$$

$$(n-m)\,\frac{m(m+3)}{2}-\frac{n(n-1)}{2}, \qquad \text{if } \frac{m(m+3)}{2} \geqq n,$$

Hence *the number of parameters of order* 3 *and of the first type is*

$$\frac{mn(n+1)(n+2)}{6}+\frac{n(n-1)}{2},$$

and of the second type

$$(n-m)\frac{m(m+1)(m+2)}{6}+\frac{m(m+3)}{2}\left\{n-\frac{(m+1)(m+2)}{4}\right\},$$

$$\textit{if } \frac{m(m+3)}{2} \leqq n,$$

and

$$(n-m)\frac{m(m+1)(m+2)}{6}+\frac{n(n-1)}{2}, \qquad \textit{if } \frac{m(m+3)}{2} \geqq n.$$

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The case $n = 2$ has not been here treated, for ŻORAWSKI, and more recently, FORSYTH,* have already discussed it in full. The methods here used apply however to this case also, and effect a considerable simplification of the work. For example for the determination of the parameters of the second type and order we have the single equation

$$2h_{11}\frac{\partial f}{\partial h_{11}}+4\Lambda_{11,11}\frac{\partial f}{\partial \Lambda_{11,11}}=0.$$

The parameter sought is therefore

$$P \equiv \frac{\Lambda_{11,11}}{h_{11}^2}$$

which is readily seen to be the square of the geodesic curvature of a curve on the surface whose squared linear element is given by the differential form under consideration.

The functions $y_{1/11}, y_{2/11}$ have a simple geometric interpretation, for the equations

$$\underset{11}{y_1}=0, \qquad \underset{11}{y_2}=0$$

are,† if z_1, the independent variable, is the length of arc of a geodesic line, the equations which determine the geodesics on the surface.

*FORSYTH, Philosophical Transactions, vol. 201 (1903), A, pp. 329–402.

†Cf. BIANCHI, *Differentialgeometrie*, p. 151.